ASYMPTOTIC BEHAVIOR OF THE MOTION OF A GAS WHEN IT FLOWS INTO A CYLINDRICAL CAVITY

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1. In [1] we considered the flow of a gas into a spherical cavity. In the present paper we investigate the asymptotic behavior of gasdynamics functions when an ideal gas flows into a cylindrical cavity.

In the rarefaction wave with center in the plane (r, t) which arises as a result of the flow of gas into a cylindrical cavity, the entropy in the principal term is constant. Therefore we take as our equation of state

$$p = A\rho^{\varkappa}. \tag{1.1}$$

As a result of the application of a transformation of the coordinates and functions which is invariant with respect to the equations of gasdynamics, we can assume that the constant $A = 1$ and that the values corresponding to the start of the flow are $r_0 = 1$ for the radius of the cavity, $t_0 = -1$ for the time, and $u = -1$ for the velocity of the free boundary. The system of gasdynamics equations in this case has the form

$$(h-1)\left[\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r}\right] + c\frac{\partial u}{\partial r} + \frac{uc}{r} = 0, \tag{1.2}$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + (h-1)c\frac{\partial c}{\partial r} = 0, \quad h = (\varkappa+1)/(\varkappa-1)$$

(u is the velocity of the gas and c is the velocity of sound).

2. The centered rarefaction wave in a neighborhood of the vertex A ($r = 1$, $t = -1$) will be plane in the principal term, and therefore the asymptotic behavior in the neighborhood can be represented as an expansion in powers of $(t + 1)$ with coefficients which depend on the quantity $\xi = (r - 1)/(t + 1)$:

$$f = f_0(\xi) + f_1(\xi)(t+1) \qquad (f = u,\ c). \tag{2.1}$$

The curves ξ = const correspond in the principal term to the α characteristics of the bundle, where the quantity ξ varies from the value $\xi = \xi_0 = -1$, corresponding to the free boundary, to the value $\xi = \xi_1 = u^0 + c^0$, corresponding to the separating characteristic [u^0, c^0 are the values of the unperturbed velocity and the velocity of sound at the point A ($r = 1$, $t = -1$)]. By substituting (2.1) into the system (1.2) and comparing the coefficients of corresponding powers of $t + 1$, we obtain

$$u = -1 + \frac{h-1}{h}(\xi+1) + \left[\frac{\xi+1}{h(2-h)} + \frac{3(\xi+1)^2}{h^2(3\varkappa-5)} + K(\xi+1)^{h/2}\right](t+1) + \ldots, \tag{2.2}$$
$$c = \frac{\xi+1}{h} + \left[\frac{\xi+1}{h(2-h)} - \frac{(\varkappa-5)(\xi+1)^2}{2h^2(3\varkappa-5)} - \frac{(\varkappa-5)(\varkappa-1)}{2(3\varkappa-1)}K(\xi+1)^{h/2}\right](t+1) + \ldots$$

At $\varkappa = 5/3$ and $\varkappa = 3 (h = 4.2)$ we find singularities. The value of the constant is determined by the initial distribution of the functions in the unperturbed gas in a neighborhood of the point A. If at $t = -1$ we have $u = u^0 + u'(r + 1)$, $c = c^0 + c'(r - 1)$, then

$$K = (hc^0)^{\frac{2-h}{2}}\left[\frac{h+2}{2h^2}u' - \frac{(h-1)(h+2)}{2h^2}c' - \frac{3(h-1)}{2h(h-4)}c^0 + \frac{u^0}{2h^2} - \frac{1}{(2-h)h}\right].$$

It is important to note that for the initial state corresponding to rest [$u^0 = u' = c' = 0$, and consequently $c^0 = 1/(h - 1)$] the value

$$K = -\frac{(\varkappa-1)^2(3\varkappa-1)}{(\varkappa-3)(3\varkappa-5)}\left(\frac{2}{\varkappa+1}\right)^{h/2},$$

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i.e., $K > 0$ for $5/3 < \varkappa < 3$ and $K < 0$ for $\varkappa > 3$.

The determination of the functions f_0, f_1 in the expansion (2.2) was based on the assumption that

$$f_1(\xi)(t+1)/f_0(\xi) \to 0 \quad \text{as} \quad t \to -1.$$

For $\varkappa < 3$ this condition is satisfied over the entire range of variation of ξ. When $\varkappa > 3(h < 2)$ and $\xi \to -1$,

$$c_1(\xi)(t+1)/c_0(\xi) \sim (\xi+1)^{h/2-1}(t+1) \to \infty.$$

Consequently for $\varkappa > 3$ the determination of the asymptotic behavior in a neighborhood of a point in the region adjacent to the free boundary, $\xi = -1$, requires further investigation, which we shall carry out below.

3. The symptotic formulas (2.2), when $1 < \varkappa < 3$, indicate the existence of a time interval $-1 \leqslant t \leqslant t_1$ during which the velocity of the free boundary is constant. The quantity ξ at the free boundary is constant at $\xi = -1$ for these values of time. Therefore the determination of the value t_1 and the asymptotic behavior in a neighborhood of the free boundary will be based on an expansion of the gasdynamics functions in powers of $\eta = \xi + 1$. Taking account of the asymptotic behavior obtained in Section 2, we can represent the desired expansion in the form

$$u = -1 + u_1(t)\eta + u_2(t)\eta^\alpha + \ldots, \quad c = c_1(t)\eta + c_2(t)\eta^\alpha + \ldots + (\varkappa \neq 5/3), \tag{3.1}$$

where $\alpha = 2$ when $1 < \varkappa < 5/3$; $\alpha = h/2$ when $5/3 < \varkappa < 3$.

The equations determining $u_1(t)$ and $c_1(t)$ are obtained in the usual way and on the assumption that $\eta/t \ll 1$ over the entire range of integration, in particular as $t \to 0$, while the initial data are obtained from the asymptotic behavior described in Sec. 1:

$$c_1'(t+1) - c_1 + \frac{1}{2}(\varkappa+1)u_1c_1 + \frac{1}{2}(\varkappa-1)\frac{t+1}{t}c_1 = 0, \tag{3.2}$$

$$u_1'(t+1) - u_1 + u_1^2 + (h-1)c_1^2 = 0, \quad 1 < \varkappa < 3;$$
$$u_1(-1) = 2/(\varkappa+1), \; c_1(-1) = (\varkappa-1)/(\varkappa+1). \tag{3.3}$$

The equations for the functions $u_2(t)$, $c_2(t)$ are:

$$(h-1)\left[c_2'(t+1) - 2c_2\right] + (2h-1)u_1c_2 + (h+1)c_1u_2 + \frac{(t+1)}{t}\left(c_2 - u_1c_2 + \frac{t+1}{t}c_1\right) = 0, \tag{3.4}$$

$$u_2'(t+1) - 2u_2 + 3u_1u_2 + 3(h-1)c_1c_2 = 0, \quad 1 < \varkappa < 5/3;$$

$$u_2 \simeq -\frac{3(\varkappa-1)^2}{(\varkappa+1)^2(3\varkappa-5)}(t+1), \quad c_2 \simeq \frac{(5-\varkappa)(\varkappa-1)^2}{2(\varkappa+1)^2(3\varkappa-5)}(t+1) \quad \text{as} \quad t \to -1; \tag{3.5}$$

$$\frac{2}{\varkappa-1}c_2'(t+1) + \left[\left(\frac{(\varkappa+1)}{(\varkappa-1)^2}+1\right)u_1 - \frac{(\varkappa+1)}{(\varkappa-1)^2}\right]c_2 + \frac{5+\varkappa}{2(\varkappa-1)}c_1u_2 + \frac{t+1}{t}c_2 = 0, \tag{3.6}$$

$$u_2'(t+1) - \frac{1}{2}\frac{\varkappa+1}{\varkappa-1}u_2 + \frac{3\varkappa-1}{2(\varkappa-1)}u_1u_2 + \frac{3\varkappa-1}{(\varkappa-1)^2}c_1c_2 = 0, \quad 5/3 < \varkappa < 3;$$

$$u_2 \simeq K(t+1), \quad c_2 = -\frac{(\varkappa-1)(\varkappa+5)}{2(3\varkappa-1)}K(t+1) \quad \text{as} \quad t \to -1. \tag{3.7}$$

<u>THEOREM.</u> If $1 < \varkappa < 2$, then the functions $u_1(t)$ and $c_1(t)$ are finite and nonzero in the interval $-1 < t < 0$. If $2 < \varkappa < 3$, then there exists a value $t_1 = t_1(\varkappa)$, $-1 < t_1 < 0$, such that $u_1(t)$ and $c_1(t)$ tend to infinity as $t \to t_1$.

Suppose that

$$c_1(t) = \frac{t+1}{t}M(t), \quad u_1(t) = \frac{1+t}{t}L(t), \tag{3.8}$$

then $M(t)$ and $L(t)$, according to (3.2), (3.3), are determined by the system

$$\frac{dM}{dL} = \frac{M[(\varkappa+1)L + (\varkappa-3)]}{2(L^2 - L + (h-1)M^2)}, \quad M = -\infty, \quad L = -\infty; \tag{3.9}$$

$$t=-\exp\left(-\int\limits_{-\infty}^{M}\frac{2dM}{M\,[(\varkappa+1)L+(\varkappa-3)]}\right); \tag{3.10a}$$

$$t=-\exp\left(-\int\limits_{-\infty}^{L}\frac{dL}{L^2-L+(h-1)M^2}\right). \tag{3.10b}$$

Since c > 0 and η > 0, it follows that M < 0. The asymptotic behavior of the desired integral curve (l) in a neighborhood of the node (M = −∞, L = −∞) has the form

$$L=\frac{2}{\varkappa-1}M+\frac{\varkappa-1}{3-\varkappa}+\frac{2-\varkappa}{2}\left(\frac{\varkappa-1}{3-\varkappa}\right)^2\frac{1}{M}+\frac{(\varkappa-1)^2(5-3\varkappa)(2-\varkappa)}{(5+\varkappa)(3-\varkappa)^3}\frac{1}{M^2}+\ldots \tag{3.11}$$

The axis M = 0 is the integral of Eq. (3.9). In the half-plane $M \leqslant 0$ the singular points of Eq. (3.9) will be:

$$O\,(L=0,\ M=0),\quad A\,(L=1,\ M=0),\quad (L=\pm\infty,\ M=-\infty),\ C\left(L=\frac{3-\varkappa}{\varkappa+1},\quad M=-\frac{\varkappa-1}{\varkappa+1}\sqrt{3-\varkappa}\right).$$

LEMMA. In the half-plane M < 0 the integral curve (l) reaches the point 0 for $1<\varkappa<2$, not intersecting at any point the zero isocline, which is the straight line (g): $L = (3-\varkappa)/(\varkappa+1)$; for $\varkappa=2$ it coincides with the line L = 2M + 1; for $2<\varkappa<3$ it intersects the line (g) below the point C without intersecting at any point the infinity isocline, which is the ellipse (f):

$$L^2-L+(h-1)M^2=0.$$

Proof of the Lemma. In the region L < 0, M < 0 the integral curve (l) lies above the hyperbola (d): $L^2-L=(h-1)^2M^2$ if $1<\varkappa<5/3$, coincides with it when $\varkappa=5/3$, and lies below it if $5/3<\varkappa<2$. This follows from the sign of the asymptotic value of the difference between the values of L for the same value of M on the curves (l) and (d) in a neighborhood of the point (L = −∞, M = −∞):

$$L(l)-L(d)\simeq(5-3\varkappa)/2(3-\varkappa)$$

and the sign of the difference between the values of the right side of Eq. (3.9) on the curve (d) and the tangent of the angle of inclination of the tangent line to the curve (d):

$$\frac{dM}{dL}-\left(\frac{dM}{dL}\right)_d=\frac{(3\varkappa-5)M}{4(L^2-L+(h-1)M^2)}.$$

The curve (l) may hit the point 0 according to one of the asymptotic formulas

$$L\simeq C\,(M)^{2/(3-\varkappa)}; \tag{3.12a}$$

$$L\simeq\frac{2}{(\varkappa-2)(\varkappa-1)}M^2. \tag{3.12b}$$

The asymptotic formula for the curve (d) in a neighborhood of O(0, 0) will be

$$L\simeq-(h-1)M^2. \tag{3.13}$$

The difference between (3.12b) and (3.13) is positive when $1<\varkappa<5/3$ and negative when $5/3<\varkappa<2$, therefore when $1<\varkappa<5/3$, the integral curve (l) lies between the integral M = 0 and the hyperbola (d) (Fig. 1) and hits the point O(0, 0) in accordance with the asymptotic formula (3.12a) for some negative value of the constant C, which depends on the index $\varkappa$: When $5/3<\varkappa<2$, the integral curve cannot hit the point O(0, 0) from the direction of M < 0, and therefore it intersects the axis L = 0 at a value M < 0. The hyperbola (h): $L^2-L=(h-1)^2M^2+(h-1)(3\varkappa-5)M/(3-\varkappa)$, before intersecting the zero isocline (g) or the infinity isocline (f), lies below the integral curve (l) when $5/3<\varkappa<2$ and above it when $2<\varkappa<3$. This follows from the sign of the asymptotic value of the difference between the values of L for the same value of M in a neighborhood of the point (L = −∞, M = −∞):

$$L(l)-L(h)\simeq\frac{(2-\varkappa)(\varkappa-1)^3(5-3\varkappa)}{8(3-\varkappa)^2(5+\varkappa)}\frac{1}{M^2}$$

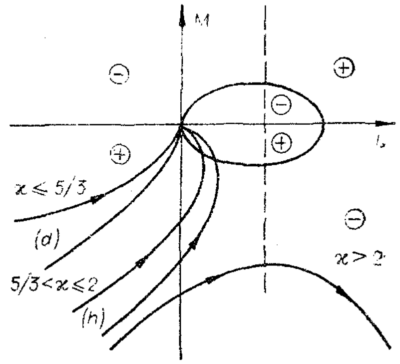

Fig. 1

and the sign of the difference between the values of the right side of Eq. (3.9) on the curve (h) and the tangent of the angle of inclination of the tangent line to the curve (h):

$$\frac{dM}{dL}-\left(\frac{dM}{dL}\right)_h=\frac{(3\varkappa-5)M^2[(\varkappa-3)L+(\varkappa-1)+(h-1)(3-\varkappa)M]}{[4(h-1)(3-\varkappa)M^2+2(3\varkappa-5)M](L^2-L+(h-1)M^2)}. \tag{3.14}$$

The sign of the difference (3.14) is positive when $\varkappa > 5/3$, while it is negative for $\varkappa < 2$. To see this, we note that the denominator of (3.14) is positive outside the infinity isocline (f), since it is greater than zero when $M < M_0 = [(5-3\varkappa)(\varkappa-1)]/[4(3-\varkappa)]$ and $M_0 > M_1 = (5-3\varkappa)/(3-\varkappa)h$ are the coordinates of the point of intersection of the curves (f) and (h). The numerator of (3.14), set equal to zero, coincides with the asymptotic equation for the hyperbola (h); therefore its sign, and consequently the sign of the difference (3.14), will be determined by substituting some point belonging to the appropriate branch of the hyperbola (h), for example $(L=0,\ M=[(3\varkappa-5)(\varkappa-1)]/2(3-\varkappa))$, and when $\varkappa > 5/3$, it will coincide with the sign of the fraction $(2-\varkappa)/(3-\varkappa)$. Furthermore, the hyperbola (h) intersects the isocline (f) at a point (L_1, M_1) lying to the left (to the right) of the zero isocline (g) if $5/3 < \varkappa < 2$ (if $2 < \varkappa < 3$). This follows from the quadratic equation determining the quantity $X = L_1 - (3-\varkappa)/(\varkappa+1)$:

$$X^2+\frac{5-3\varkappa}{\varkappa+1}X+\frac{8(\varkappa-2)}{h^2(3-\varkappa)^2}=0.$$

Consequently when $5/3 < \varkappa < 2$, the integral curve (l), after intersecting the axis L = 0, will intersect the infinity isocline (f), and according to the scheme of isoclines (see Fig. 1), it will hit the point O(0, 0). Since L > 0, the asymptotic behavior will be given by (3.12a). When $\varkappa = 2$, the integral (l) coincides with the straight line L = 2M + 1. When $2 < \varkappa < 3$, the integral curve (l) will not intersect the infinity isocline (f) but will intersect the zero isocline (g), after which it will go off to the point ($L = \infty$, $M = -\infty$), since the point of intersection of the hyperbola (h) with the line (g) lies to the left of the point of intersection of the hyperbola (h) with the ellipse (f), while the integral curve (l) lies below the hyperbola (h) (scheme of isoclines shown in Fig. 1).

Proof of the Theorem. For $1 < \varkappa < 2$ the curve (l) does not intersect the zero isocline (g) anywhere as M varies from $-\infty$ to 0. According to the quadrature (3.10a), along the curve (l) the quantity t varies monotonically from -1 to 0, while as $M \to 0$, the quantity $t \sim M^{2/(3-\varkappa)}$ or $M \sim t^{(3-\varkappa)/2}$, and by virtue of the asymptotic formula (3.12a), the quantity $L \sim t$. Consequently, by (3.8), the functions $u_1(t)$ and $c_1(t)$ remain finite in the interval $-1 < t < 0$, and for t = 0 the function $u_1(t)$ remains finite, while $c_1(t)$ becomes infinite:

$$c_1(t) \sim t^{(1-\varkappa)/2}. \tag{3.15}$$

For $\varkappa = 2$, the functions $u_1(t)$ and $c_1(t)$ can be written explicitly as follows:

$$u_1(t) = (1+t)(2t-1)/3t,\ c_1(t) = (t+1)^3/3t.$$

For $2 < \varkappa < 3$ the curve (l) does not intersect the infinity isocline anywhere, and therefore the integral under the exponential sign in the quadrature (3.10b) increases constantly, remaining finite as L varies from $-\infty$ to $+\infty$. Consequently the value of t increases monotonically from -1 to $t_1 < 0$. As $t \to t_1$, the functions L and M tend to infinity according to the asymptotic formulas:

$$L \simeq -\frac{2}{\varkappa+1}\frac{t_1}{t_1-t}, \quad M \simeq \frac{\varkappa-1}{\varkappa+1}\frac{t_1}{t_1-t}.$$

The functions $u_1(t)$ and $c_1(t)$ correspondingly tend to infinity according to the asymptotic formulas:

$$u_1(t) \simeq -\frac{2}{\varkappa+1}\frac{1+t_1}{t_1-t}, \quad c_1(t) \simeq \frac{\varkappa-1}{\varkappa+1}\frac{1+t_1}{t_1-t}. \tag{3.16}$$

Remark. The function $u_1(t)$ is monotone decreasing and the function $c_1(t)$ is monotone increasing in the interval $-1 < t < t_1$. To see this, we note that for $t = -1$ we have $u_1'(t) > 0$ and $c_1'(t) > 0$. The derivative $c_1'(t)$ cannot vanish when $t = t_0$ if $u'(t_0) < 0$ for $t < t_0$, since we then have $c_1(t_0) > 0$ and $c_1''(t_0) = -\frac{1}{2}(\varkappa+1)u_1'(t_0)c_1(t_0) + \frac{\varkappa-1}{2t_0^2}c_1(t_0) > 0$.

In an analogous manner, $u_1'(t)$ cannot vanish when $t = t_{00}$ if $c_1'(t) < 0$ for $t \leqslant t_{00}$, since in this case

$$u_1''(t_{00}) = -\frac{4}{\varkappa+1}c_1(t_{00})c_1'(t_{00}) < 0.$$

Investigation of the Functions $u_2(t)$, $c_2(t)$. The functions $u_2(t)$, $c_2(t)$ are the solutions of systems of linear equations whose coefficients and right sides are continuous functions of t, $u_1(t)$, and $c_1(t)$. Consequently $u_2(t)$ and $c_2(t)$ are finite in the same intervals as $u_1(t)$ and $c_1(t)$. From the system (3.4) and the asymptotic formula (3.15) it follows that for $1 < \varkappa < 5/3$ and $t \to 0$ we have $u_2(t) \sim t^{1-\varkappa}$, $c_2(t) \sim t^{-(1+\varkappa)/2}$. From the system (3.6), (3.7) and the asymptotic formula (3.15) it follows that for $5/3 < \varkappa < 2$ and $t \to 0$, $u_2(t)$ is finite and $c_2(t) \sim t^{(1-\varkappa)/2}$. The system (3.5) is invariant with respect to the transformation $u_2 = KV_2$, $c_2 = KW_2$. Here the functions $V_2(t)$ and $W_2(t)$ are independent of the value of K. Making use of the monotonicity of the functions $u_1(t)$ and $c_1(t)$, as in the preceding remark, we can prove that the function $V_2(t)$ is monotone increasing and the function $W_2(t)$ is monotone decreasing in the interval $-1 < t < t_1$. From the system (3.6), (3.7) and the asymptotic formula (3.16) it follows that as $t \to t_1$,

$$V_2 \simeq L(h-1)(t_1-t)^{-(h+2)/2}, \; W_2 \simeq -L(t_1-t)^{-(h+2)/2}, \\ L > 0$$

and since $V_2(t)$ and $W_2(t)$ are monotonic, the constant $L > 0$. Consequently the asymptotic behavior of the functions $u_2(t)$ and $c_2(t)$ as $t \to t_1$ can be given by the formulas

$$u_2 \simeq KL(t_1-t)^{-(h+2)/2}, \quad c_2 \simeq -\frac{1}{(h-1)}KL(t_1-t)^{-(h+2)/2}, \quad L > 0.$$

Equations (3.2) were obtained on the assumption that η/t is small. If we also take account of the ratio of the latter to the earlier terms, then in the time interval $-1 < t < t_1(\varkappa)$ the formulas (3.1) obtained earlier for the asymptotic behavior in a neighborhood of the free boundary can be used only if η/t is small when $1 < \varkappa < 2$ and $\eta^{h-2}(t_1-t)^{-h}$ is small when $2 < \varkappa < 3$.

4. In order to find the asymptotic behavior for $1 < \varkappa < 2$ in a complete neighborhood of the point 0 ($r = 0$, $t = 0$), the gasdynamics functions can be represented in the form

$$u = u(s), \; c = r^{(3-\varkappa)/2}c(s), \text{ where } s = (r+t)/t. \tag{4.1}$$

This representation is based on the asymptotic behavior found earlier for a neighborhood of the free boundary in this range of $\varkappa$ as $r \to 0$, $t \to 0$:

$$u \simeq -1, \quad c \simeq -D\left(\frac{r+t}{t}\right)r^{(3-\varkappa)/2}, \tag{4.2}$$

and its validity is bounded by the condition that η/t is small, which if $r \to 0$ and $t \to 0$ is equivalent to the condition that s is small. The value of s varies from $s = 0$ on the free boundary to $s = -\infty$ as $t \to -0$ and from $s = +\infty$ as $t \to +0$ to $s = 1$ ($r = 0$, $t > 0$). As $r \to 0$, the equations determining the functions $u(s)$ and $c(s)$ have the form

$$u' = 0, \; c'[(s-1)^2 + (s-1)] = -c. \tag{4.3}$$

Taking account of the asymptotic behavior as $s \to 0$ [this follows from (4.2)], we see that the solution of (4.3) will be

$$u = -1,\ c = Ds/(s-1). \tag{4.4}$$

The solution thus obtained is not valid in the entire neighborhood of the point 0, since on the axis for t > 0, according to (4.1), (4.4), $c \sim r^{(1-\varkappa)/2} \to \infty$. There arises a shock wave moving from the reflection axis. Since ahead of the front we have $\rho \sim r^{(3-\varkappa)/(\varkappa-1)}$, $u \simeq -1$, we must look for the solution behind the front in the form

$$u = u(s),\ \rho = r^{(3-\varkappa)/(\varkappa-1)}\,\rho(s),\ p = r^{(3-\varkappa)/(\varkappa-1)}p(s) \to c = c(s). \tag{4.5}$$

It can be seen that the shock wave will be of infinite intensity and the flow behind the front will not be isentropic. The entropy quantity can be represented in the form

$$A = r^{\varkappa-3}A(s).$$

The functions u(s), c(s), A(s) are determined by the system of equations:

$$\begin{gathered}(s-1)[u+(s-1)]A' = (3-\varkappa)uA,\\ 2c'[u(s-1)-(s-1)^2]+(\varkappa-1)(s-1)cu'+(\varkappa-1)uc = 0,\\ \varkappa(u-s+1)[(\varkappa-1)(s-1)(u-s+1)u'+2cc'] - (3-\varkappa)(s-1)c^2 = 0\end{gathered} \tag{4.6}$$

and the boundary conditions on the front $s = s_b$ and on the axis s = 1:

$$u(s_b) = -1+\frac{h-1}{h}s_b,\quad c(s_b) = \frac{\sqrt{h+1}}{h}s_b,\quad u(1)=0,\quad c(1) = \text{const}\neq 0. \tag{4.7}$$

The system (4.6) has the first integral

$$A^{2/(3-\varkappa)}c^{2/(\varkappa-1)}\left(\frac{u}{s-1}-1\right) = \text{const}.$$

Making use of this, by means of the substitution

$$c^2 = L(s-1)^2,\ u = (s-1)(K+1) \tag{4.8}$$

we reduce the system (4.6) to an equation and a quadrature:

$$\frac{dL}{dK} = \frac{L[(1-\varkappa)\varkappa(K^2+LK)-(3-\varkappa)L-\varkappa(\varkappa+1)K(K^2-L)](\varkappa-1)}{K[(K^2+LK)\varkappa(\varkappa-1)+(3-\varkappa)L-(\varkappa-1)\varkappa(2+K)(K^2-L)]}; \tag{4.9}$$

$$\frac{dK}{ds} = \frac{L[2\varkappa(\varkappa-1)(K+1)+(3-\varkappa)]-\varkappa(\varkappa-1)(K+1)K^2}{\varkappa(\varkappa-1)(K^2-L)(s-1)}. \tag{4.10}$$

According to (4.7), (4.8), L(1) = ∞, and from (4.10) it follows that $K(1) = -(1+(h-2)/2\varkappa) = K_0$. The point ($L = \infty$, $K = K_0$) is a saddle-type singular point of Eq. (4.9), and the desired integral curve coincides with its separatrix, which emanates from this point in the direction of

$$L = \frac{\varkappa K_0^3(K_0+1)}{[4\varkappa K_0-(3-\varkappa)](K-K_0)}$$

until it intersects the parabola $L = (h+1)K^2$, on which lies the point corresponding to the shock wave. The quadrature (4.10) gives us the value of $s = s_b$ corresponding to the wave front. On the front of the shock wave the values of u and c will be finite, and the pressure and density will be of the order of $r^{(3-\varkappa)/(\varkappa-1)}$. On the axis $(r=0,\ t>0)$ $u=0$, $p \sim t^{(3-\varkappa)/(\varkappa-1)}$, $A(s)=\infty$, $\rho = 0$.

5. For $2<\varkappa<3$ the asymptotic representation of the functions in a neighborhood of the point on the free boundary ($t = t_1$, $r = r(t_1)$) is taken in the form

$$c = \frac{1+t_1}{h}(t_1-t)^{2/(h-2)}z^{h/(h-2)}f_1(z),\quad u = -1-\frac{h-1}{h}(1+t_1)\,(t_1-t)^{2/(h-2)}f_2(z),\quad z = \eta^{(h-2)/h}/(t_1-t). \tag{5.1}$$

This representation is based on the asymptotic behavior found in Sec. 3 in a neighborhood of the free boundary and the condition for its applicability. The equations determining the functions $f_1(z)$ and $f_2(z)$ have the form

$$\frac{df_1}{df_2} = \frac{(h-2)f_1^2+(h-1)^2(2-h)f_2^2+h(2h^2-4h+4)f_2-h^3}{(h-1)^2\left[(2-h)f_2^3+2hf_2^2\right]+(h-2)f_1^2f_2+2hf_1^2-h^2f_2}; \tag{5.2}$$

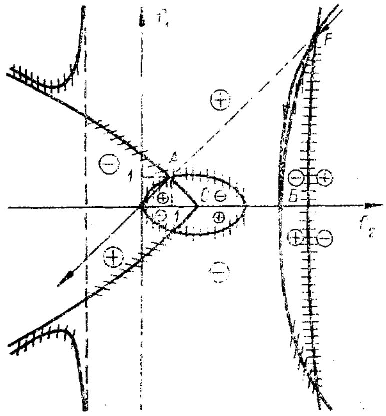

Fig. 2

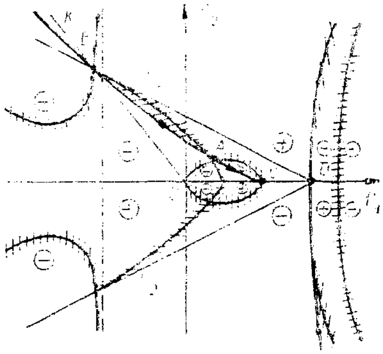

Fig. 3

$$z\frac{df_1}{dz} = -f_1\frac{h(h-2)f_1^2 + h(h-1)^2(2-h)f_2^2 + h^2(2h^2-4h+4)f_2 - h^4}{(h-2)^2 f_1^2 - [h^2 + (h-1)(2-h)f_2]^2}. \tag{5.3}$$

When $t < t_1$, the value of z corresponding to the free boundary is $z = +0$, and according to the asymptotic behavior found in Sec. 3, as $z \to +0$, we have

$$f_1(z) \simeq 1 - \frac{KLh}{1+t_1}z^{h/2}, \quad f_2(z) = 1 - \frac{KLh}{1+t_1}z^{h/2}. \tag{5.4}$$

The desired solution will be an integral curve which has the asymptotic behavior (5.4) as $z \to +0$ and satisfies the conditions on the free boundary for some value of $z > 0$. The system written out above has an exact solution with the asymptotic formula (5.4) as $z \to 0$:

$$f_1 = f_2, \quad f_1(f_1 - 1) = \left(\frac{LKh}{1+t_1}\right)^{-2/h}\frac{1}{z},$$

which coincides with the desired solution on some interval of the values of z. The desired solution is determined from an investigation of Eq. (5.2). The scheme of isoclines is shown in Fig. 2. If the functions $f_1(z)$ and $f_2(z)$ are to be single-valued, the integral curve (l) of Eq. (5.2) must not intersect the straight lines $L_{1,2}$ outside of the singular points of Eq. (5.3):

$$(L_{1,2})\, f_1 = \pm\frac{1}{2-h}[h^2 + (h-1)(2-h)f_2].$$

The zero isoclines of Eq. (5.2) will be the straight line $f_2 = 0$ and the hyperbola

$$f_1^2 = (h-1)^2(f_2 - f_1^-)(f_2 - f_1^+), \quad f_1^{\pm} = \frac{h\left(h^2 - 2h + 2 \pm \sqrt{3h^2 - 6h + 4}\right)}{(h-1)^2(2-h)}.$$

The infinity isocline will be the curve

$$f_1^2 = \frac{(h-1)^2(h-2)}{2h + (h-2)f_2}f_2(f_2 - f_2^-)(f_2 - f_2^+), \quad f_2^- = \frac{h}{h-1}, \quad f_2^+ = \frac{h^2}{(h-2)(h-1)}.$$

It can be seen that

$$f_1^+ > f_1^-, \quad f_2^+ > f_2^-, \quad f_1^+ > f_2^+, \quad f_1^- < f_2^-.$$

The singular points of Eq. (5.2) will be:

$$A(f_1 = 1,\ f_2 = 1),\quad B(f_1 = 0,\ f_2 = f_2^+),\quad C(f_1 = 0,\ f_2 = f_2^-),\quad F\left(f_1 = \frac{h^2}{(2-h)^2},\ f_2 = \frac{h^2}{(2-h)^2}\right).$$

Further investigation will depend on the sign of K.

A. K > 0. The desired integral curve of Eq. (5.2) joins the points A and F by a separate "whisker" of the node F — the integral $f_1 = f_2 (-\infty \leqslant f_1 \leqslant 1,\ h^2/(h-2)^2 = f_{1F} \leqslant f_1 < \infty)$ — and joins the points F and B by the separatrix of the saddle point B, which is different from the integral $f_1 = 0$ (see Fig. 2). To the value $f_1 = 0$ corresponds $z = \infty$, i.e., $t = t_1$. The integral curve has a break-point at the node B; this is admissible, since the curve corresponding to the point B is the characteristic of the initial gasdynamics system. The equation of the free boundary when $t_1 < t < t_1 + \varepsilon$ and ε is sufficiently small in the principal term will be $z = z_B$, where z_B is the value of z corresponding to the point B. The fact that the corresponding conditions will be satisfied on the free boundary follows from the asymptotic formulas for $z \to z_B$:

$$f_1^2 \approx -\frac{4h^4 (z - z_B)}{(h-2)^3 (h-1) z_B},\quad f_2 \approx \frac{h^2}{(h-1)(h-2)} - \frac{2h^2 (3h-2)}{(h^2-1)(h-2)^2}(z - z_B).$$

The asymptotic behavior of the gasdynamics functions in a neighborhood of a point on the free boundary $(r_1 = r(t_1),\ t_1)$ is given by the following formulas:

$$u \approx -1 - \frac{h}{h-2}(1+t_1)\,|z_B|^{h/(h-2)}\,(t_1 - t)^{2/(h-2)} + \frac{2h(3h-2)}{(h+1)(h-2)^2}(1+t_1)\,|z_B|^{h/(h-2)}(t - t_1)^{2/(h-2)}\left(\frac{z}{z_B} - 1\right),$$

$$c \approx \frac{2h}{h-2}(1+t_1)\,|z_B|^{(h+2)/2(h-2)}\,(t - t_1)^{2/(h-2)}\sqrt{\frac{z_B - z}{(h-2)(h-1)}}.$$

Thus, along the free boundary $z = z_B$ for $t > t_1$ the velocity begins to increase.

B. K < 0. In this case, according to the asymptotic formula (5.4), the motion along the integral curve takes place in the direction of increasing f_1. When the point F $\left(f_1 = f_2 = \frac{h}{h-2}\right)$ is reached, the functions $f_1(z)$ and $f_2(z)$ are no longer single-valued, since this point, not being a singular point of Eq. (5.3), lies on its infinity isocline, the straight line L_2. The β characteristics intersect and have in the plane (r, t) an envelope $z = z_F$, where z_F is the value of z corresponding to the point F. Consequently there arises a shock wave which goes out to the free boundary at time $t = t_1$ and gives it an infinite acceleration.

6. When $\varkappa > 3$, the asymptotic equation (2.2) obtained in Sec. 2, as $\xi \to -1$, $t \to -1$, will be valid only in the part where the values of $\eta = (\xi + 1)(1 + t)^{2/(h-2)}$ are sufficiently large, and as $\eta \to \infty$, in the principal terms, it will have the form

$$u \simeq -1 + \frac{h-1}{h}(1+\xi)\left[1 + \frac{\varkappa+1}{2}K\eta^{(h-2)/2}\right],\quad c \simeq \frac{1}{h}(\xi+1)\left[1 - \frac{(\varkappa+1)(\varkappa+5)}{2(3\varkappa-1)}K\eta^{(h-2)/2}\right]. \tag{6.1}$$

Taking account of formula (6.1), we can represent the asymptotic behavior in a complete neighborhood of the point (r = 1, t = −1) as

$$u \simeq -1 + \frac{h-1}{h}(\xi+1) f_1(\eta),\quad c \simeq \frac{1}{h}(\xi+1) f_2(\eta) \tag{6.2}$$

for $(\xi + 1) \to 0$ and finite values of η. According to (6.1) as $\eta \to \infty$, we have

$$f_1(\eta) \simeq 1 + \frac{\varkappa+1}{2}K\eta^{(h-2)/2},\quad f_2(\eta) = 1 - \frac{(\varkappa+1)(\varkappa+5)}{2(3\varkappa-1)}K\eta^{(h-2)/2}, \tag{6.3}$$

and the functions $f_1(\eta)$ and $f_2(\eta)$ must satisfy the equations

$$\frac{df_2}{df_1} = -f_2\frac{(4-h)h^2 - (2-h)f_2^2 - h(-2h^2 + 8h - 4)f_1 + (2-h)(h-1)^2 f_1^2}{2hf_2^2 + (2-h)f_1 f_2^2 + 2h(h-1)(3-h)f_1^2 + (h-1)^2(h-2)f_1^3 - h^2(4-h)f_1}; \tag{6.4}$$

$$\eta \frac{df_2}{d\eta} = -f_2 \frac{(h-2)\{(4-h)h^2-(2-h)f_2^2+h(2h^2-8h+4)f_1+(2-h)(h-1)^2 f_1^2\}}{(2-h)^2 f_2^2-[(4-h)h+(h-2)(h-1)f_1]^2}, \qquad (6.5)$$

the asymptotic equation (6.3), and for some value $\eta = \eta_1$ the conditions on the free boundary. As in Sec. 5, the solution of the problem is concentrated in the determination of the integral curve of Eq. (6.4), $f_2 = f_2(f_1)$, joining the points $f_1 = f_2 = 1$ with the point $f_1(\eta_1)$, $f_2(\eta_1)$, corresponding to the free boundary. In Fig. 3 we show the scheme of zero and infinity isoclines of Eq. (6.4), the singular points of this equation, its integral — the straight line (K): $f_2 = -f_1$ — and the infinity isoclines of Eq. (6.5), the straight lines $L_{1,2}: f_2 = \pm \frac{1}{2-h}[(h-2)(h-1)f_1 + (4-h)h]$. By virtue of the uniqueness, the desired integral curve (l) can intersect the straight line (K) only at the singular points of Eq. (6.4), and if the solution is to be single-valued, it intersects the straight lines ($L_{1,2}$) only at the singular points of Eq. (6.5). According to (6.2), the function $f_2(\eta) \geqslant 0$ when $\eta \geqslant 0$. The zero isoclines will be the integral $f_2 = 0$ and the hyperbola

$$f_2^2 = (h-1)^2(f_1 - f_{10})(f_2 - f_{20}), \text{ where } f_{i0} = \frac{h\left(-h^2+4h-2 \mp \sqrt{5-(h-3)^2}\right)}{(h-1)^2(2-h)} \quad (i = 1, 2),$$

and the infinity isocline will be the curve

$$f_2^2 = \frac{(f_1 - f_{1\infty})(f_2 - f_{2\infty})f_1}{h(2-h)f_1 + 2} h(2-h), \text{ where } f_{1\infty} = \frac{h}{h-1}, \quad f_{2\infty} = \frac{h(4-h)}{(h-1)(2-h)},$$

with $f_{10} < f_{1\infty} < f_{2\infty} < f_{20}$. The singular points of Eq. (6.4) are: $A(f_1 = f_2 = 1)$, a saddle point; $B\left(f_1 = \frac{-h(4-h)}{(2-h)^2}, \quad f_2 = \frac{h(4-h)}{(2-h)^2}\right)$, a node; $C\left(f_1 = \frac{h}{h-1}, \quad f_2 = 0\right)$, a node; $D\left(f_1 = \frac{4-h}{2-h}\frac{h}{h-1}, \quad f_2 = 0\right)$, a saddle point; $E\,(f_1 = \pm\infty, \quad f_2 = \mp\infty)$, dicritical nodes; and $O(f_1 = f_2 = 0)$, a node.

The relative positions of points and curves shown in Fig. 3 are obtained by some simple calculations. According to the asymptotic formula (6.3), the integral curve (l) emanates from the saddle point A along its separatrix in the direction

$$df_2/df_1 = -(\varkappa + 5)/(3\varkappa - 1),$$

where for $K > 0$ it goes in the direction of increasing f_1 and for $K < 0$ it goes in the direction of decreasing f_1.

A. $\underline{K > 0.}$ The separatrix of the saddle point A, coinciding with the curve (l), will inevitably reach the point C, since $f_2 = 0$ is the integral of Eq. (6.4). According to the quadrature (6.5) when f_2 varies from 1 to 0, the value of η will vary from ∞ to 0, and as $\eta \to 0$, the function $f_2 \sim \eta^{-2/(h-1)(h-2)}$, while $f_1 \simeq h/(h-1)$. The curve $\eta = 0$ corresponds to the free boundary. This follows from the representation (6.2) and the above-described asymptotic behavior as $\eta \to 0$.

B. $\underline{K < 0.}$ The desired integral curve of Eq. (6.4) from the saddle point A along its separatrix, going in the direction of decreasing f, reaches the node B. From the node B, passing through the infinitely remote dicritical node E, the integral curve moves along the separatrix of the saddle point D, distinct from the integral $f_2 = 0$, and reaches the point D (see Fig. 3). The resulting breakpoint in the integral curve at the node B is admissible, since the point B corresponds to the characteristic of the initial gasdynamics system. The equation of the free boundary for $-1 < t < -1 + \varepsilon$ and sufficiently small ε in the principal term will be $\eta = \eta_D$, $\eta_D < 0$, where η_D is the value of η corresponding to the point D. The fact that the corresponding conditions are satisfied on the free boundary follows from the asymptotic formulas obtained in a neighborhood of the point D and the value of η_D:

$$f_2 \approx -\sqrt{\frac{(h+1)(4-h)}{2+h}}\left(f_1 - \frac{(4-h)h}{(2-h)(h-1)}\right), \quad f_2^2 \approx \frac{-4(4-h)h^2}{(2-h)^2(h-1)\eta_D}(\eta - \eta_D).$$

From the asymptotic behavior of the resulting equation for the free boundary in the plane rt,

$$r \approx -t + \eta_D(1 + t)^{(4-h)/(2-h)},$$

it follows that the motion of the free boundary when t > −1 is accelerated.

The main result of the investigation is the following: The velocity of the free boundary remains constant in the time interval (t_0, t_1), where time t_0 corresponds to the beginning of the flow and t_1 coincides with the time t_f when the free boundary reaches the axis of the cylinder if the adiabatic index $\varkappa \leqslant 2$, and $t_1 < t_f$ if $\varkappa > 2$. In particular, when $\varkappa > 3$, the time t_1 coincides with the time t_0 if the gas was at rest before the flow began.

LITERATURE CITED

1. Ya. M. Kazhdan, "Asymptotic behavior of a convergent rarefaction wave," in: Proceedings of the Section for Numerical Methods in Gasdynamics, Second International Colloquium on the Gasdynamics of Explosions and Reactive Systems, Vol. 3 [in Russian], Vses. Tsentr. Akad. Nauk SSSR, Moscow (1971).

EXTREMAL VALUES OF CYLINDER DRAG BEHIND A DISK IN SUPERSONIC FLOW

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UDC 533.601.1

The authors examine axisymmetric supersonic flow over a cylinder of diameter D, ahead of which is mounted a disk of diameter d < D on a thin connecting piece of length l. The flow separates on the disk, and near the body surface there is a flow circulation region, separated from the external flow by a mixing zone which spans a certain "dividing stream area" originating from the disk edge and incident on the end of the cylinder.

Taking account of the special features of the flow investigated, one can judge that the best procedure is to seek a solution of the problem based on a system of exact Navier–Stokes equations. However, with all the promise of this type of approach, even when adequately efficient numerical algorithms are available, a solution to the Navier–Stokes equations for a compressible fluid has been obtained as yet only for low and medium Reynolds numbers. An alternative approach is to construct an adequate mathematical model which would describe, as far as possible, the main characteristic features of the flow investigated.

As such an approximate model we choose a numerical model in which a result is obtained by applying the "large particle" numerical method [1, 2] to the equations describing the motion of an ideal gas — it reproduces the separated flow over the body in the process of establishing the solution corresponding to steady flow. The ideal fluid model has been used in a number of papers in investigating separated flows, including that at the front of a spiked body (cf. [3, 4]). Among the factors governing the fruitfulness of using this computational model the main one is evidently that it reproduces reliably the basic elements of the flow outside the circulation zone. The shape and dimensions of this zone are determined largely by the geometry of the components. Here we locate a large-scale unit vortex, separated from the walls and the outer flow by a comparatively thin viscous layer in which the transverse pressure gradient is small and which does not appreciably affect the pressure distribution on the body surface, at least above a certain Reynolds number (from Re $\geqslant$ 500, according to the data of [5]). One would expect that for the body of the composition considered here, with d < D, $l \sim$ D, the local Reynolds number for the flow in the circulation region will be large enough [5]. On the other hand, it is known that computational schemes for ideal gas flows similar to [1, 2], because of their inherent computational "dissipation" properties, give results with features characteristic of large Reynolds number flow. Finally, one should take into account the known idea that the base pressure depends only slightly on Re [6] in supersonic flow at large Reynolds number.

These considerations support the expectation that the numerical model will in the main correctly reflect the actual fluid flow in the entire computed region. The results then ob-

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